Kuwait University | Math 101 | May 27, 2001
Dept of Math & Comp Sci | Final Examination | Duration: 2 hours

Answer the following questions. Calculators, Phones and Pagers are not allowed.

Eeach question is worth 4 points

1. Evaluate each of the following limits, if it exists.

   (a) $\lim\limits_{x\to\infty} \sqrt[3]{\dfrac{1-8x^3}{x\,(x^2+1)}}$

   (b) $\lim\limits_{x\to 0} \dfrac{\sin 2x - 2\sin x}{x \sin x}$

2. Let $f(x) = 2 + \sqrt[3]{x^2-1}$.

   (a) Show that the graph of f has a vertical tangent at the point $(1,2)$.

   (b) Does the graph of f have a cusp at the point $(-1,2)$?

3. Let $y = \sqrt[3]{2t^2+t-9}$ and $t = \cot 2x + \csc^2 2x - 4$. Find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{8}$.

4. Let $f(x) = \begin{cases} \cos x & \text{, if } x < 0, \\ 1 - x^3 & \text{, if } x \geqslant 0. \end{cases}$. Use the definition of the derivative to find $f'(0)$.

5. Sand is falling into a conical pile at a rate of $2\,\mathrm{m}^3/\mathrm{sec}$. The height of the cone is always $\frac{2}{3}$ the radius of its base. Find the rate of change of the radius of the pile when it contains $48\pi\,\mathrm{m}^3$ of sand.

6. Evaluate the following integrals.

   (a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos x\, dx$

   (b) $\displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$

Let $f(x) = \dfrac{1}{\sqrt{2x+1}}$.

   (a) Find the average value of f on $[4,12]$.

   (b) Find a number z that satisfies the conclusion of the Mean Value Theorem for Definite Integrals.

8. Let $f(x) = \displaystyle\int_1^{x^3+x} \sqrt{7+t^4}\, dt + \int_3^5 \cos^7 w\, dw$. Show that f is an increasing function on $\mathbb{R}$.

9. Find the arc length of the graph of the equation $y = \frac{3}{2}x^{\frac{2}{3}}$ from $A(1, \frac{3}{2})$ to $B(8,6)$.

10. Let R be the region bounded by the curves $y = 2x - x^2$ and $y = 0$.

   (a) Find the area of R.

   (b) Set up an integral that can be used to find the volume of the solid obtained by revolving the region R about the line $x = 5$.

Final Exam. Solution Key Math. 101 May 27, 2001

1. (a) $\lim\limits_{x\to\infty}\sqrt[3]{\dfrac{1-8x^3}{x(x^2+1)}}=\sqrt[3]{\lim\limits_{x\to\infty}\dfrac{1-8x^3}{x^3+x}}=\sqrt[3]{\lim\limits_{x\to\infty}\dfrac{\frac{1}{x^3}-8}{1+\frac{1}{x^2}}}=\sqrt[3]{-8}=\boxed{-2}.$

   (b) $\lim\limits_{x\to 0}\dfrac{\sin 2x-2\sin x}{x\sin x}=\lim\limits_{x\to 0}\dfrac{2\sin x\cos x-2\sin x}{x\sin x}=2\lim\limits_{x\to 0}\dfrac{\cos x-1}{x}=\boxed{0}.$

2. $f'(x)=\dfrac{2x}{3(x^2-1)^{\frac{2}{3}}}.$

   (a) <u>Yes</u>, f is continuous at $x=1$ and $\boxed{\lim\limits_{x\to 1^+}f'(x)=\lim\limits_{x\to 1^-}f'(x)=\infty}.$

   (b) <u>No</u>, f is continuous at $x=-1$ But, $\boxed{\lim\limits_{x\to -1^+}f'(x)=\lim\limits_{x\to -1^-}f'(x)=-\infty}.$

3. $\dfrac{dy}{dt}=\dfrac{4t+1}{3(2t^2+t-9)^{\frac{2}{3}}},\quad \dfrac{dt}{dx}=-2\csc^2 2x-4\csc^2 2x\cot 2x.$

   $\frac{\pi}{8})=\boxed{-1},\quad \left.\dfrac{dy}{dt}\right|_{t=-1}=\boxed{-\frac{1}{4}},\quad \left.\dfrac{dt}{dx}\right|_{x=\frac{\pi}{8}}=\boxed{-12},\quad \left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{8}}=\left.\dfrac{dy}{dt}\right|_{t=-1}\times\left.\dfrac{dt}{dx}\right|_{x=\frac{\pi}{8}}=\boxed{3}.$

4. $\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{(1-x^3)-1}{x}=0,\quad \lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{\cos x-1}{x}=0\Rightarrow f'(0)=\boxed{0}.$

5. $V=\frac{1}{3}\pi r^2h=\frac{2\pi}{9}r^3,\quad 2=\dfrac{dV}{dt}=\frac{2\pi}{3}r^2\dfrac{dr}{dt},\qquad \left.\dfrac{dr}{dt}\right|_{V=48\pi}=\boxed{\dfrac{1}{12\pi}\text{ m/sec}}.$ $[r=6\,m,\text{ when } V=48\pi\text{ m}^3].$

2. (a) Put $u=\sin x,\quad du=\cos x\,dx,\quad \displaystyle\int_0^{\frac{\pi}{2}}(\sin x)^5\cos x\,dx=\int_0^1 u^5du=\boxed{\frac{1}{6}}.$

   (b) Put $u=x^3+1,\quad du=3x^2dx,\quad \displaystyle\int x^2(x^3+1)^{-1/2}dx=\frac{1}{3}\int u^{-1/2}du=\frac{2}{3}u^{\frac{1}{2}}+C=\frac{2}{3}\sqrt{x^3+1}+C.$

3. (a) $f_{av}=\dfrac{1}{12-4}\displaystyle\int_4^{12}\frac{1}{\sqrt{2x+1}}\,dx=\boxed{\frac{1}{4}}.$ [Put $u=2x+1$, $du=2dx$]

   (b) $\frac{1}{4}=f(z)=\dfrac{1}{\sqrt{2z+1}}.$ Solving for z gives $z=\dfrac{15}{2}=\boxed{7.5}$ which is in $(4,12)$.

8. $f'(x)=(3x^2+1)\sqrt{7+(x^3+x)^4}>0,\quad \forall x\in\mathbb{R}\Rightarrow f$ is increasing on $\mathbb{R}$.

9. $y'=x^{-\frac{1}{3}},\quad L_1^8=\displaystyle\int_1^8\sqrt{1+(y')^2}dx=\int_1^8 x^{-\frac{1}{3}}\sqrt{1+x^{\frac{2}{3}}}dx=\boxed{\sqrt{125}-\sqrt{8}}.$ [Put $u=1+x^{\frac{2}{3}},du=\frac{2}{3}x^{-\frac{1}{3}}dx$].

10. $y=2x-x^2=1-(x-1)^2$

   (a) Area of $R=\displaystyle\int_0^2(2x-x^2)\,dx=\boxed{\frac{4}{3}}.$

   (b) By Cylindrical Shells: Volume of shell $=2\pi(5-x)(2x-x^2)\,dx$

   Volume of the solid of revolution $=2\pi\displaystyle\int_0^2(5-x)(2x-x^2)\,dx$